Proceedings of Basic and Applied Sciences
ISSN 1857-8179 (paper). ISSN 1857-8187 (online). http://www.anglisticum.mk
Proceedings of the 1st International Conference on New Horizons in Basic and Applied Science, Hurghada – Egypt, Vol 1(1), 2013.

Common Fixed Point Theorems for Weak Contraction Conditions of Integral Type

R. A. Rashwan[1], H. A. Hammad[2]
[1]Department of Mathematics, Faculty of Science, Assuit University, Assuit 71516, Egypt,
[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt,
E-mail: rr_rashwan54@yahoo.com; E-mail: h_elmagd89@yahoo.com

Received November 13, 2013; Revised November 14, 2013; Accepted November 15, 2013

Abstract: *In this paper we shall establish two common fixed point theorems for a contractive condition and A-contraction mappings of integral type which improve and extend the results of P.B. Prajapati, R. Bhardwaj [14], M. O. Olatinwo [13] and many others.*

Keywords: *Common fixed point, general contractive mapps of integral type, weak contraction, metric spaces.*

Introduction

An elementary account of the Banach contraction principle and some applications, including its role in solving nonlinear ordinary differential equations,was given in [9]. The contraction mapping theorem is used to prove the inverse function theorem as in [20]. A beautiful application of contraction mappings to the construction of fractals is in [21]. After the classical result by Banach, Kannan [12] gave a substantially new Contractive mapping to prove the fixed point theorem. Since then there have been many theorems emerged as generalizations under vario1us contractive conditions. Such conditions involve linear and nonlinear expressions. The interested reader who wants to know more about this matter is recommended to go deep into the survey articles by Rhoades ([16], [17], [18]) and Bianchini [7], and into the references there in.

Theorem 1.1. [14] Let (X,d) be a complete metric space, $c\in(0,1)$ and $f:X\to X$ be a mapping such that for each $x,y\in X$,

$$d(fx,fy)\le cd(x,y), \tag{1.1}$$

then f has a unique fixed point $z\in X$, such that for each $x\in X$, $\lim_{n\to\infty} f^n x = z$.

In 2002, A. Branciari[8] analysed the existence of fixed point for mapping T defined on a complete metric space (X,d) satisfying a general contractive condition of integral type in the following theorem:

Theorem 1.2 [14] Let (X,d) be a complete metric space, $c\in(0,1)$ and $f: X\to X$ be a mapping such that for each $x,y\in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt, \tag{1.2}$$

where $\varphi : R^{+} \to R^{+}$ is a Lebesgue-integrable mapping which is summable on each compact subset of R^{+} non negative and such that for each ε > 0, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then f has a unique fixed point $z \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$. Rhoades [19] extending the result of Branciari by replacing the condition (1.2) by the following

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{\max\{d(x,y),d(x,fx),\mathrm{d}(y,fy),\frac{[d(x,fx)+d(y,fx)]}{2}\}} \varphi(t)dt \quad (1.3)$$

for each $x, y \in X$, with some $c \in (0,1)$.

Literature abounds with several generalizations of the classical Banach's fixed point theorem since 1922. For some of these generalizations of the classical Banach's fixed point theorem and various contractive definitions that have been employed, we refer the readers to [1, 2, 3, 4, 5, 6, 10, 11, 16, 22].

Akram et al.[13] introduced a new class of contraction maps, called A-contraction, which is proper super class of Kannan's [12] , Bianchini's [7] and Reich's [15] type contractions, as follows:

Let a non- empty set A consisting of all $\alpha : R_{+}^{3} \to R$ functions satisfying

(A1): α is continuous on the set R_{+}^{3} of all triplets of non-negative reals. (With respect to the Euclidean metric on R^3).

(A2): $a \le kb$ for some $k \in [0,1)$ whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$) or $a \le \alpha(b,b,a)$ for all a, b.

Definition 1.3. A self maps f and g on a metric space X are said to be A-contraction if it satisfies the condition $d(fx, gy) \le \alpha(d(x, y), d(fx, x), (y, gy))$

(1.4)

for each $x, y \in X$, with some $\alpha \in A$.

Example 1.4. Let f **,** g be self -mappings on a metric space (X,d) satisfying:

$d(fx,gy) \le \beta \max\{d(fx,x)+d(gy,y),$
$d(gy,y)+d(x,y),\ d(fx,x)+d(x,y)\}$ For each $x, y \in X$, with some $\beta \in [0,\frac{1}{2})$, is an A-contraction.

Definition 1.5. A self maps f and g on a metric space X are called a weak contraction or (δ, L)−weak contraction if and only if there exist two constants, $\delta \in [0, 1)$ and $L \geq 0$, such that

$$d(fx, gy) \leq \delta d(x, y) + Ld(y, gx) \tag{1.5}$$

Definition 1.6. [13] A function $\psi : R^+ \to R^+$ is called a comparison function if it satisfies the following conditions: (i) ψ is monotone increasing; (ii) $\lim_{n\to\infty} \psi^n(t) = 0, \ \forall\, t \geq 0$ (iii) $\psi(0) = 0$.

For two mappings f, g we define:

Definition 1.7. A self maps f and g on a metric space X are called a (L, ψ)−weak contraction of integral type if and only if there exist a constant L ≥ 0 and a continuous comparison function $\psi : R^+ \to R^+$ such that $\forall\, x, y \in X$,

$$\int_0^{d(fx,gy)} \varphi(t)d\nu(t) \leq L\left(\int_0^{d(x,fx)} \varphi(t)d\nu(t)\right)^r \left(\int_0^{d(y,gx)} \varphi(t)d\nu(t)\right) + \psi\left(\int_0^{d(x,y)} \varphi(t)d\nu(t)\right) \tag{1.6}$$

where $r \geq 0$, $\varphi : R^+ \to R^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative such that for each ε > 0, $\int_0^{\varepsilon} \varphi(t)d\nu(t) > 0$ and $\nu : R^+ \to R^+$ is a monotone increasing function.

Definition 1.8. A self maps f and g on a metric space X are called a (ϕ , ψ)−weak contraction of integral type if and only if there exist a continuous monotone increasing function $\phi : R^+ \to R^+$ such that ϕ (0)=0 and a continuous comparison function $\psi : R^+ \to R^+$ such that $\forall\, x, y \in X$,

$$\int_0^{d(fx,gy)} \varphi(t)d\nu(t) \leq \phi\left(\int_0^{d(x,fx)} \varphi(t)d\nu(t)\right) \left(\int_0^{d(y,gx)} \varphi(t)d\nu(t)\right) + \psi\left(\int_0^{d(x,y)} \varphi(t)d\nu(t)\right) \tag{1.7}$$

Where $\varphi : R^{+} \to R^{+}$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative such that for each ε > 0, $\int_0^{\varepsilon} \varphi(t)d\nu(t) > 0$ and $\nu : R^{+} \to R^{+}$ is also monotone increasing.

Remark.1.9

(i) The contractive conditions (1.6) and (1.7) do not require any additional conditions for the uniqueness of the common fixed points of f and g. This is an improvement on the results of M. O. Olatinwo [13] which leads to Berinde [5]. (ii)The contractive condition (1.6) reduces to (1.5) if r = 0, ν (t) = t, $\varphi(t) = 1$, $t \in R^{+}$ and $\psi(u) = \delta u \quad \forall$ u ∈ R^{+}.

In this paper, we will prove two theorems, the first theorem is a generalization and extension of the results of P.B.prajapati, R. Bhardwaj [14], the second theorems improve and extend the results of M. O. Olatinwo [13]

Main Results

Theorem 2.1. Let f and g be a self- mappings of a complete metric space (X, d) satisfying the following condition**:**

$$\int_0^{d(fx,gy)} \varphi(t)dt \le \alpha\Big(\int_0^{d(x,y)} \varphi(t)dt,\ +\beta\Big(\int_0^{d(x,y)+d(y,gx)} \varphi(t)dt$$
$$\int_0^{d(x,fx)} \varphi(t)dt,\ \int_0^{d(y,gy)} \varphi(t)dt\Big) \qquad ,\int_0^{d(x,fx)+d(y,gx)} \varphi(t)dt,\ \int_0^{\max\{d(x,fy),d(y,gx)\}} \varphi(t)dt\Big), \tag{2.1.1}$$

for each $x, y \in X$, and for some $\alpha \in A$. Where $\varphi : R^{+} \to R^{+}$ is a Lebesgue-integrable mapping which is summable on each compact subset of R^{+}, non negative and such thatfor each ε > 0, $\int_0^{\varepsilon} \varphi(t)dt > 0$.

(2.1.2)

Then f and g have a unique common fixed point.

Proof: Let $x_\circ \in X$ be an arbitrary point of X. Define $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. Then by (2.1.1), we have

$$\int_0^{d(x_{2n+1}, gx_{2n+2})} \varphi(t)dt = \int_0^{d(fx_{2n}, gx_{2n+1})} \varphi(t)dt$$

$$\le \alpha\left(\int_0^{d(x_{2n}, x_{2n+1})} \varphi(t)dt, \int_0^{d(x_{2n}, fx_{2n})} \varphi(t)dt,\right.$$

$$\left.\int_0^{d(x_{2n+1}, gx_{2n+1})} \varphi(t)dt\right)$$

$$+\beta\left(\int_0^{d(x_{2n}, x_{2n+1})+d(x_{2n+1}, gx_{2n})} \varphi(t)dt,\right.$$

$$\int_0^{d(x_{2n}, fx_{2n})+d(x_{2n+1}, gx_{2n})} \varphi(t)dt,$$

$$\left.\int_0^{\max\{d(x_{2n}, fx_{2n+1}), d(x_{2n+1}, gx_{2n})} \varphi(t)dt\right)$$

$$\le \alpha\left(\int_0^{d(x_{2n}, x_{2n+1})} \varphi(t)dt,\right.$$

$$\left.\int_0^{d(x_{2n}, x_{2n+1})} \varphi(t)dt, \int_0^{d(x_{2n+1}, x_{2n+2})} \varphi(t)dt\right)$$

$$+\beta\left(\int_0^{d(x_{2n}, x_{2n+1})+d(x_{2n+1}, x_{2n+1})} \varphi(t)dt,\right.$$

$$\int_0^{d(x_{2n}, x_{2n+1})+d(x_{2n+1}, x_{2n+1})} \varphi(t)dt,$$

$$\left.\int_0^{\max\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\}} \varphi(t)dt\right)$$

$$\leq \alpha\Big(\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt, \int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt, \int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\Big)$$

$$+\beta\Big(\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt, \int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt, \int_0^{d(x_{2n},x_{2n+2})}\varphi(t)dt\Big).$$

Then by axiom (A2) of the function α, $\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt \leq K\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt + L\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt$,

(2.1.3)

for some $K, L \in [0,1))$ with k + L∈ [0, 1) as α∈ A. In this fashion, one can deduce that

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \leq K\int_0^{d(x_{n-1},x_n)}\varphi(t)dt \leq (K+L)^2\int_0^{d(x_{n-2},x_{n-1})}\varphi(t)dt \leq ....$$

$$+L\int_0^{d(x_{n-1},x_n)}\varphi(t)dt \leq (K+L)\int_0^{d(x_{n-1},x_n)}\varphi(t)dt \leq (K+L)^n\int_0^{d(x_\circ,x_1)}\varphi(t)dt.$$

Taking the limit as $n\to\infty$, we get $\lim_{n\to\infty}\int_0^{d(x_n,x_{n+1})}\varphi(t)dt = 0$ as $K+L\in(0,1)$, From (2.1.3) we have,

$$\lim_{n\to\infty} d(x_{2n},x_{2n+1}) = 0$$

(2.1.4)

Now we establish $\{x_n\}$ is a Cauchy sequence. It is sufficient to prove that $\{x_{2n}\}$ is a Cauchy. Suppose the contrary. There exists $\varepsilon>0$ and subsequences $\{x_{2m(k)}\}$, $\{x_{2n(k)}\}$ such that $k<2m(k)<2n(k)$ with $d(x_{2m(k)},x_{2n(k)})\geq\varepsilon$,

$$d(x_{2m(k)},x_{2n(k)-1})<\varepsilon \qquad (2.1.5)$$

Selection and peer-review under responsibility of ICNHBAS, 2013 http://www.nhbas2013.com

Now, $d(x_{2m(k)-1}, x_{2n(k)-1}) \le d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2n(k)-1}) < d(x_{2m(k)-1}, x_{2m(k)}) + \varepsilon$. (2.1.6)

So by (2.1.4). (2.1.6) and taking the limit we get

$$\lim_{n\to\infty} \int_0^{d(x_{2m(k)-1}, x_{2n(k)-1})} \varphi(t)dt \le \int_0^{\varepsilon} \varphi(t)dt. \tag{2.1.7}$$

From (2.1.3), (2.1.5) and (2.1.7) we have

$$\int_0^{\varepsilon} \varphi(t)dt \le \int_0^{d(x_{2m(k)}, x_{2n(k)})} \varphi(t)dt \le (K+L) \int_0^{d(x_{2m(k)-1}, x_{2n(k)-1})} \varphi(t)dt \le (K+L)\int_0^{\varepsilon} \varphi(t)dt,$$

which is a contradiction, since $K+L \in (0,1)$.

Since (X,d) be a complete metric space. Then $\{x_{2n}\}$ converges to some $z \in X$, that is $\lim_{n\to\infty} x_{2n} = z$.

By using (2.1.1) we have $\int_0^{d(x_{2n+1}, gz)} \varphi(t)dt = \int_0^{d(fx_{2n}, gz)} \varphi(t)dt$

$$\le \alpha\Big(\int_0^{d(x_{2n},z)} \varphi(t)dt, \int_0^{d(x_{2n},fx_{2n})} \varphi(t)dt, \int_0^{d(z,gz)} \varphi(t)dt\Big)$$

$$+\beta\Big(\int_0^{d(x_{2n},z)+d(z,gx_{2n})} \varphi(t)dt, \int_0^{d(x_{2n},fx_{2n})+d(z,gx_{2n})} \varphi(t)dt, \int_0^{\max\{d(x_{2n},fz), d(z,gx_{2n})\}} \varphi(t)dt\Big)$$

$$\le \alpha\Big(\int_0^{d(x_{2n},z)} \varphi(t)dt, \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt, \int_0^{d(z,gz)} \varphi(t)dt\Big)$$

$$\beta(\int_0^{d(x_{2n},z)+d(z,x_{2n+1})}\varphi(t)dt,$$

$$\int_0^{d(x_{2n},x_{2n+1})+d(z,x_{2n+1})}\varphi(t)dt,$$

$$\int_0^{\max\{\mathrm{d}(x_{2n},fz),d(z,x_{2n+1})\}}\varphi(t)dt).$$

$$\int_0^{d(z,gz)}\varphi(t)dt \le \alpha(0,0,\int_0^{d(z,gz)}\varphi(t)dt)$$

Taking the limit as $n \to \infty$, we get $+\beta(0,0,\int_0^{d(z,fz)}\varphi(t)dt)$

Using the axiom (A2) of the function α, one gets $\int_0^{d(z,gz)}\varphi(t)dt \le K.0 + B.0$, i. e.,

$d(z,gz)=0$ or $z = gz$. By the same way, one can prove that $fz = z$. Then f and g have a common fixed point $z \in X$. To prove the uniqueness .Suppose that $w \neq (z)$ be another common fixed point of f and g. Then from (2.1.1), we obtain

$$\int_0^{d(z,w)}\varphi(t)dt = \int_0^{d(fz,gw)}\varphi(t)dt \le \alpha(\int_0^{d(z,w)}\varphi(t)dt,$$

$$\int_0^{d(z,fz)}\varphi(t)dt, \int_0^{d(w,gw)}\varphi(t)dt)$$

$$+\beta(\int_0^{d(z,w)+d(w,gz)}\varphi(t)dt, \int_0^{d(z,fz)+d(w,gz)}\varphi(t)dt,$$

$$\int_0^{\max\{d(z,fw)+d(w,gz)\}}\varphi(t)dt)$$

$$\int_0^{d(z,w)}\varphi(t)dt \le$$

$$\alpha(\int_0^{d(z,w)}\varphi(t)dt, \int_0^{d(z,z)}\varphi(t)dt, \int_0^{d(w,w)}\varphi(t)dt)$$

Selection and peer-review under responsibility of ICNHBAS, 2013 http://www.nhbas2013.com

$$+\beta\left(\int_0^{2d(z,w)}\varphi(t)dt, \int_0^{d(z,z)+d(w,z)}\varphi(t)dt,\right.$$

$$\left.\int_0^{\max\{d(z,w)+d(w,z)\}}\varphi(t)dt\right) \qquad \int_0^{d(z,w)}\varphi(t)dt \le \alpha\left(\int_0^{d(z,w)}\varphi(t)dt\right),0,0)$$

$$+\beta\left(\int_0^{2d(z,w)}\varphi(t)dt, \int_0^{d(w,z)}\varphi(t)dt, \int_0^{d(w,z)}\varphi(t)dt\right).$$

So by axiom (A2) of the function α, $\int_0^{d(z,w)}\varphi(t)dt \le K.0 + L.\int_0^{d(z,w)}\varphi(t)dt.$,

which is a contradiction since $L \in [0,1)$. Then $w = z$ and so f and g have a unique common fixed point. Now, we give an example satisfying all requirements of Theorem 2.1.1.

Example 2.2 Let $X = [0,1]$ be a metric space with usual metric $d(x, y) = |x - y|$.

Define $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{5}, & 0 \le x < \frac{1}{4} \\ x, & \frac{1}{4} \le x \le 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{5}, & 0 \le x < \frac{1}{4} \\ 1-4x, & \frac{1}{4} \le x \le 1 \end{cases}$$

Let $\varphi(t) = 1$, $x = 0$ and $y = \frac{1}{5}$

It is clearly that $\varphi(t) = 1$ is a Lebesgue-integrable mapping which is summable on each compact subset of R^+, non negative and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon}\varphi(t)dt = \int_0^{\varepsilon}dt = \varepsilon > 0$$

Also from the inequality (2.1.1) we have $\int_0^{d(fx,gy)}\varphi(t)dt = \int_0^{d(f0,g\frac{1}{5})}dt = \int_0^{d(0,\frac{1}{5})}dt = \frac{1}{5}$

(2.2.1)

$$\alpha\left(\int_0^{d(x,y)}\varphi(t)dt, \int_0^{d(x,fx)}\varphi(t)dt, \int_0^{d(y,gy)}\varphi(t)dt\right) + \beta\left(\int_0^{d(x,y)+d(y,gx)}\varphi(t)dt, \int_0^{d(x,fx)+d(y,gx)}\varphi(t)dt, \int_0^{\max\{d(x,fy),d(y,gx)\}}\varphi(t)dt\right)$$

$$=\alpha\left(\int_0^{d(0,\frac{1}{5})} dt, \int_0^{d(0,f0)} dt, \int_0^{d(\frac{1}{5},g\frac{1}{5})} dt\right) + \beta\left(\int_0^{d(0,\frac{1}{5})+d(\frac{1}{5},g0)} dt, \int_0^{d(0,f0)+d(\frac{1}{5},g0)} dt, \int_0^{\max\{d(0,f\frac{1}{5}),d(\frac{1}{5},g0)\}} dt\right) =$$

$\alpha(\int_0^{\frac{1}{5}} dt,0,0) \ +\beta(\int_0^{1} dt, \int_0^{\frac{4}{5}} dt, \int_0^{\frac{4}{5}} dt)$. So by axiom (A2) of the function α,

$$\leq K.0+L.\int_0^{\frac{4}{5}} dt=\frac{4L}{5} \tag{2.2.2}$$

From (2.2.1) and (2.2.2) we have $\frac{1}{5}\leq\frac{4L}{5}$. This is valid if $L=\frac{1}{4}<1$. In this example f and g are continuous at $x=\frac{1}{5}$ and $\frac{1}{5}$ is a unique common fixed point of the mappings f and g.

Theorem 2.3. Let (X,d) be a complete metric space and $f,g:X\to X$ be (ϕ,ψ)-weak contraction of integral type (1.7). Suppose that $\psi:R^+\to R^+$ is a continuous comparison function and ν, ϕ : $R^+\to R^+$ are monotone increasing functions such that ϕ is continuous and $\phi(0)$ = 0. Let φ: $R^+\to R^+$ be a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative, and such that for each ε > 0, $\int_0^{\varepsilon}\varphi(t)dt>0$ where ν: $R^+\to R^+$ is also an increasing function. Then f *and* g have a unique common fixed point.

Proof: Let $x_\circ \in X$ be arbitrary and let us define $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$.

Then by (1.7) we have

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)d\nu(t) = \int_0^{d(fx_{2n},gx_{2n+1})} \varphi(t)d\nu(t) \le \psi\left(\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)d\nu(t)\right)$$

$$+\phi\left(\int_0^{d(x_{2n},fx_{2n})} \varphi(t)d\nu(t)\right)\left(\int_0^{d(x_{2n+1},gx_{2n})} \varphi(t)d\nu(t)\right)$$

$$=\psi\left(\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)d\nu(t)\right)$$

$$+\phi\left(\int_0^{d(x_{2n},fx_{2n})} \varphi(t)d\nu(t)\right)\left(\int_0^{d(x_{2n+1},x_{2n+1})} \varphi(t)d\nu(t)\right)$$

$$\le \psi\left(\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)d\nu(t)\right).$$

(2.3.1)

By the same way, one can find, for all $n \ge 0$ $\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)d\nu(t) \le \psi\left(\int_0^{d(x_{2n-1},x_{2n})} \varphi(t)d\nu(t)\right)$

(2.3.2)

From (2.3.1) and (2.3.2) we deduce that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)d\nu(t) \le \ldots\ldots\ldots\ldots\ldots \le \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)d\nu(t)\right) .$$

(2.3.3)

Taking the limit in (2.3.3) as $n \to \infty$ yields $\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dv(t) = 0$

(2.3.4)

Since $\lim_{n \to \infty} \psi^n(\int_0^{d(x_\circ,x_1)} \varphi(t)dv(t)) = 0$ and for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dv(t) > 0$.

Therefore, it follows from (2.3.4) that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$

(2.3.5)

Now we establish that $\{x_n\}$ is a Cauchy sequence. It is sufficient to prove that $\{x_{2n}\}$ is a Cauchy. Suppose the contrary. There exists $\varepsilon > 0$ and subsequences $\{x_{2m(k)}\}$, $\{x_{2n(k)}\}$ such that $k < 2m(k) < 2n(k)$ with $d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon$, $d(x_{2m(k)}, x_{2n(k)-1}) < \varepsilon$

(2.3.6)

$$\int_0^{d(x_{2m(k)},x_{2n(k)})} \varphi(t)dv(t) = \int_0^{d(fx_{2m(k)-1},gx_{2n(k)-1})} \varphi(t)dv(t)$$

Again by using (1.7) we get $\leq \psi(\int_0^{d(x_{2m(k)-1},x_{2n(k)-1})} \varphi(t)dv(t))$

$$+\phi(\int_0^{d(x_{2m(k)-1},fx_{2m(k)-1})} \varphi(t)dv(t)) \int_0^{\varepsilon} \varphi(t)dv(t) \leq \int_0^{d(x_{2m(k)},x_{2n(k)})} \varphi(t)dv(t)$$

$$(\int_0^{d(x_{2n(k)-1},gx_{2m(k)-1})} \varphi(t)v(t)) \quad \leq \psi(\int_0^{\varepsilon} \varphi(t)dv(t) < \int_0^{\varepsilon} \varphi(t)dv(t)$$

(2.3.9)

which is a contraction. Therefore, we must have that $\int_0^{\varepsilon} \varphi(t)dv(t) = 0$ that is, therefore $\{x_{2n}\}$ is a Cauchy sequence .and hence convergent. Since (X,d) is a complete metric space, $\{x_{2n}\}$ convergent to some point $z \in X$, that is, $\lim_{n\to\infty} x_{2n} = z$. Again from (1.7) we

$$\int_0^{d(x_{2n+1},gz)} \varphi(t)d\nu(t) = \int_0^{d(fx_{2n},gz)} \varphi(t)d\nu(t)$$

get $\leq \psi(\int_0^{d(x_{2n},z)} \varphi(t)d\nu(t)) + \phi(\int_0^{d(x_{2n},fx_{2n})} \varphi(t)d\nu(t))(\int_0^{d(z,gx_{2n})} \varphi(t)\nu(t))$.

(2.3.10)

Taking the limit in (2.3.10) we get $\int_0^{d(z,gz)} \varphi(t)d\nu(t) \leq \psi(0) = 0$

(2.3.11)

So that from (2.3.11) we have again, a contradiction. Therefore, by the condition on φ, we have $d(z,gz)=0$ or $z=gz$. By the same way, one can prove that $z=fz$. Next: suppose that ($w \neq z$) be another common fixed point of f *and* g, then from (1.7) we have

$$\int_0^{d(z,w)} \varphi(t)d\nu(t) = \int_0^{d(fz,gw)} \varphi(t)d\nu(t)$$

$$\leq \psi(\int_0^{d(z,w)} \varphi(t)d\nu(t)) + \phi(\int_0^{d(z,fz)} \varphi(t)d\nu(t))(\int_0^{d(w,gz)} \varphi(t)\nu(t))$$

$$\leq \psi(\int_0^{d(z,w)} \varphi(t)d\nu(t)) < \int_0^{d(z,w)} \varphi(t)d\nu(t)$$

Leading to a contraction again. Therefore by the condition on φ, we have $\int_0^{d(z,w)} \varphi(t)d\nu(t) = 0$, from which it follows that $d(z,w)=0$ or $z=w$, hence, f and g have a unique common fixed point. The following example satisfying all requirements of Theorem 2.3.

Example 2.4. Let $X=[0,1]$ be a metric space with usual metric $d(x,y)=|x-y|$.

Define $f,g: X \to X$ by $fx = \begin{cases} \frac{x^2}{3}, & 0 \leq x < 1 \\ \frac{1}{2} & x=1 \end{cases}$ and

$gx = \begin{cases} \frac{x}{3}, & 0 \leq x < 1 \\ \frac{x^2}{2}, & x=1 \end{cases}$.

If we take $\nu(t)=t$, $\phi(t)=t^2$, $\psi(t)=\frac{1}{3}t$, $\varphi(t)=1$, $x=0$ and $y=\frac{1}{3}$. Then it is clearly that f and g are not continuous mappings, $\varphi(t)=1$ is a Lebesgue-integrable mapping which is summable on each compact subset of R^+, non negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\varphi(t)dt=\int_0^{\varepsilon}dt=\varepsilon>0$ Also $\nu(t)=t$ is monotone increasing function, $\phi(t)=t^2$ is monotone increasing function such that ϕ is continuous and ϕ (0) = 0, as well as $\psi(t)=\frac{1}{3}t$ is a continuous comparison function and $\varphi(t)=1$ be a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative. From (1.7) we have

$$+\phi(\int_0^{d(x_{2m(k)-1},x_{2m(k)})}\varphi(t)d\nu(t))$$

$$=\psi(\int_0^{d(x_{2m(k)-1},x_{2n(k)-1})}\varphi(t)d\nu(t)) \qquad (\int_0^{d(x_{2n(k)-1},x_{2m(k)})}\varphi(t)\nu(t))$$

(2.3.7)

$$d(x_{2m(k)-1},x_{2n(k)-1})<d(x_{2m(k)-1},x_{2m(k)})$$

Using triangle inequality we get $+d(x_{2m(k)},x_{2n(k)-1})$

$$d(x_{2m(k)-1},x_{2n(k)-1})<d(x_{2m(k)-1},x_{2m(k)})$$
$$+\varepsilon\ \to\varepsilon \text{ as } k\to\infty \tag{2.3.8}$$

Taking the limit as $k\to\infty$ in (2.3.7), and using (2.3.5), (2.3.6) and (2.3.8) we have

$$\int_0^{d(fx,gy)}\varphi(t)d\nu(t)=$$

$$\int_0^{d(f0,g\frac{1}{3})}\varphi(t)dt=\int_0^{d(0,\frac{1}{9})}dt=\frac{1}{9}\cdot$$

$$\phi(\int_0^{d(x,fx)}\varphi(t)d\nu(t))(\int_0^{d(y,gx)}\varphi(t)d\nu(t)) \quad \phi(\int_0^{d(0,f0)}\varphi(t)dt)$$

$$+\psi(\int_0^{d(x,y)}\varphi(t)d\nu(t)) \qquad =(\int_0^{d(\frac{1}{3},g0)}\varphi(t)dt)+\psi(\int_0^{d(0,\frac{1}{3})}\varphi(t)dt))$$

$= \phi(0)(\int_0^{d(\frac{1}{3},0)} dt) + \psi(\int_0^{d(0,\frac{1}{3})} dt) = \psi(\int_0^{\frac{1}{3}} dt) = \frac{1}{9}$. Then the inequality of (1.7) is valid and the point 0 is a unique common fixed point of f and g.

Theorems 2.5 Let (X,d) be a complete metric space and $f,g: X \to X$ be (L,ψ)-weak contraction of integral type. Suppose that $\psi: R^+ \to R^+$ is a continuous comparison function and $\nu : R^+ \to R^+$ monotone increasing functions. Let $\varphi: R^+ \to R^+$ be a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$, where $\nu: R^+ \to R^+$ is also an increasing function. Then f *and* g have a unique common fixed point.

Proof. The prove is similar to the proof of Theorems 2.3.

Remark 2.6.

(i) If we put $f = g$ in Theorem 2.1. Then we obtain (Theorem 2.1, [14]).
(ii) In Theorem 2.3. If we put $f = g$. Then we get (Theorem 2.1, [13]).

References

1. R. P. Agarwal, M. Meehan and D. O'Regan, (2001). Fixed Point Theory and Applications, Cambridge University.
2. R. P. Agarwal, M. Meehan and D. O'Regan, (2004). Fixed Point Theory and Applications, Cambridge University.
3. V. Berinde, (1999). A priori and A posteriori Error Estimates for a Class of ϕ-Contractions, Bulletins for Applied & Computing Math. 183-192.
4. V. Berinde, (2002). Iterative Approximation of Fixed Points, Editura Efemeride.
5. V. Berinde, (2004). Approximating Fixed Points of Weak Contractions Using Picard
6. Iteration, Nonlinear Analysis Forum 9 (1) 43-53.
7. M. Berinde and V. Berinde, (2007), On a General Class of Multi-valued Weakly Picard Mappings, J. Math. Anal. Appl. 326, 772-782.
8. R. Bianchini, (1972). Su un problema dis.Reich Riguardante la Teori dei Punti Fissi, Boll Un. Math. Ital. 5, 103-108.Branciari, (2002). A Fixed Point Theorem for Mappings Satisfying a GeneralContractive Condition of integral type, Int. J.Math. Math. Sci, 29, no.9 531-536
9. V. Bryant, (1985). Metric Space Iteration and Application, Cambridge Univ. Press, Cambridge.

ISSN 1857-8179 (paper). ISSN 1857-8187 (online). http://www.anglisticum.mk
Proceedings of the 1st International Conference on New Horizons in Basic and Applied Science, Hurghada – Egypt, Vol 1(1), 2013.

10. Lj. B. Ciric, (1971).Generalized Contractions and Fixed Point Theorems, Publ. Inst. Math. (Beograd) (N. S.) 12 (26), 19-26.
11. Lj. B. Ciric, (1974), A Generalization of Banach's Contraction Principle, Proc. Amer. Math. Soc. 45, 267-273.
12. R. Kannan, (1968). Some Results on Fixed points, Bull. Calcutta Math. Soc., 60, 71-76.
13. M. O. Olatinwo, (2010). Some Fixed Point Theorems for Weak Contraction Condition of Integral Type, Acta Universitatis Apulensis, pp. 331-338.
14. P.B. Prajapati, R. Bhardwaj, (2013). Fixed Point Theorems for A-contraction Mappings of Integral Type, South Asian J Math, 3(1), 44-48.
15. S. Reich, (1971). Kannan's fixed point theorem, Boll. Uni. Math. Ital. 4 (1971), 1-11.
16. B.E.Rhoades, (1977). A Comparison of Various Definitions of Contractive Type,
17. Trans. Amir Math. Soc. 220, 257-290.
18. B.E.Rhoades, (1982). Contractive Definitions Revisited, Topological Methods in
19. Nonlinear Functional Analysis, Contemp. Math. Vol.21, American.
20. B.E.Rhoades, (1987), Contractive Definitions, Nonlinear Analysis, World Science Publishing Singapore, 513-526.

Selection and peer-review under responsibility of ICNHBAS, 2013 http://www.nhbas2013.com